

Correlation functions of critical loop models

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Exactly solved interacting Conformal Field Theories are scarce. In 2D, with only conformal symmetry:

- Minimal models¹
- Liouville theory²

Techniques:

- Lagrangian/path integral approach
- Hamiltonian/operator approach
- Bootstrap approach

¹Dotsenko and Fateev 1985.

²Dorn and Otto 1994; Alexander B. Zamolodchikov and Alexei B. Zamolodchikov 1996.

Plan of the talk

Spin and loop models

Spectrum

Sphere 3 point functions

Sphere four-point and torus one-point functions

Conclusion

Spin and loop models

Spectrum

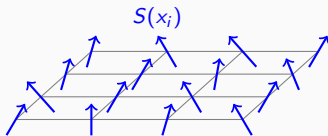
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Spin models

Atoms in a solid: interacting spins on a lattice:



Gibbs probability distribution:

$$\mathbb{P}(S) = \frac{1}{Z} e^{-\beta E(S)}, \quad Z = \int dS e^{-\beta E(S)}$$

$O(n)$, $PSU(n)$ and Potts models

- $O(n)$ model: $S(x_i) \in \mathbb{R}^n, |S(x_i)| = 1$,

$$E(S) = -J \sum_{\langle x_i, x_j \rangle} \sum_{a=1}^n S^a(x_i) S^a(x_j)$$

$\langle x_i, x_j \rangle =$ nearest neighbours

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- Potts model: $S(x_i) \in \{1, \dots, Q\}$,

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Correlation functions

Goal: compute correlation functions

$$\left\langle \prod_{i=1}^N S^{a_i}(x_i) \right\rangle = \frac{1}{Z} \int dS \left(\prod_{i=1}^N S^{a_i}(x_i) \right) e^{-\beta E(S)}$$

Examples:

- One-point function:

$$\langle S^a(x) \rangle = 0$$

- at the critical point

$$\langle S^a(x) S^b(y) \rangle \propto \frac{\delta^{a,b}}{|x - y|^{\Delta + \bar{\Delta}}}$$

Loop models

High-temperature expansion: $e^{\beta JS_i \cdot S_j} \simeq 1 + \beta JS_i \cdot S_j$

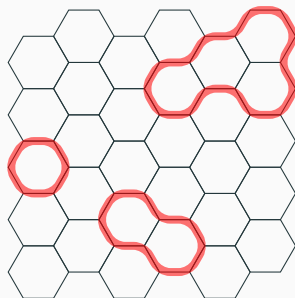
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$$Z = \int dS e^{-\beta E(S)} = \sum_{A \in \mathcal{L}(E)} (\beta J)^{|A|} n^{\ell(A)}$$

- $\mathcal{L}(E)$: configurations of **non-intersecting** loops
- $|A|$: number of red edges
- $\ell(A)$: number of loops



$$|A| = 32, \ell(A) = 3$$

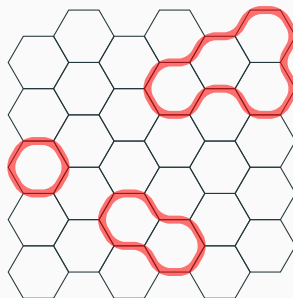
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$$n \in \mathbb{C}$$



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Correlation functions

High-temperature expansion for correlation functions:

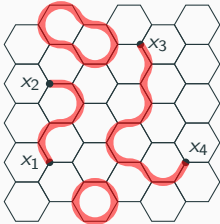
$$\left\langle \prod_{i=1}^N S^{a_i}(x_i) \right\rangle = \sum_{p \in \text{pairings}(\{x_i\})} \underbrace{\left(\prod_{(ij) \in p} \delta^{a_i, a_j} \right)}_{\text{tensorial part}} \underbrace{\sum_{A \in \mathcal{L}^p(E)} (\beta J)^{|A|} n^{\ell(A)}}_{\text{geometric part}}$$

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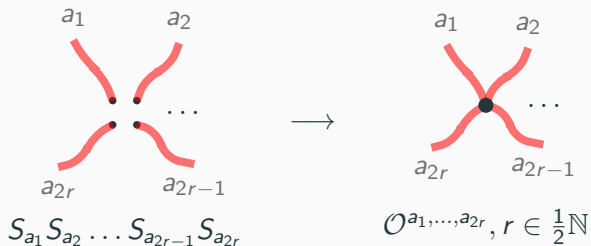
Example:

$$\langle S^{a_1}(x_1) S^{a_2}(x_2) S^{a_3}(x_3) S^{a_4}(x_4) \rangle = \delta^{a_1, a_2} \delta^{a_3, a_4} + \dots$$




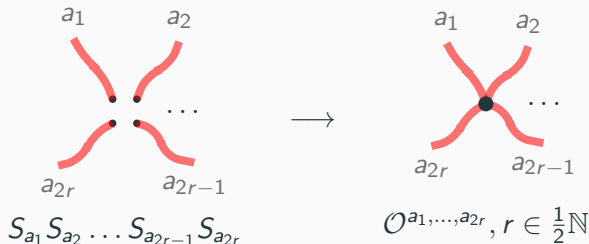
Combinatorial maps

Continuum limit:



Combinatorial maps

Continuum limit:



Rotation covariant operators:

$$V_{(r,s)}^{a_1, \dots, a_{2r}} = \sum_{k \in \mathbb{Z}_{2r}} e^{-i\pi ks} \mathcal{O}^{a_1+k, \dots, a_{2r}+k} \quad r \in \frac{1}{2}\mathbb{N}, s \in \frac{1}{r}\mathbb{Z}$$

Combinatorial maps

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$$\left\langle \prod_{i=1}^N V_{(r_i,s_i)}^{a_1^i,\dots,a_{2r_i}^i}(x_i) \right\rangle = \sum_{M \in \mathcal{M}(\{r_i\}, \{x_i\})} (\text{tensorial part}) \underbrace{\left\langle \prod_{i=1}^N V_{(r_i,s_i)}(x_i) \right\rangle^M}_{\text{geometric part}}$$

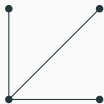
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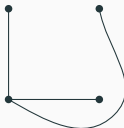
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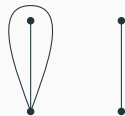
Example $\left\langle V_{(\frac{3}{2}, s)} V_{(\frac{1}{2}, 0)} V_{(\frac{1}{2}, 0)} V_{(\frac{1}{2}, 0)} \right\rangle^{\text{sphere}} :$



$$V^{abc} V^a V^b V^c$$



$$V^{acb} V^a V^b V^c$$



$$V^{aba} V^b V^c V^c$$

Critical limit

Second order phase transition at β_c : correlation length $\rightarrow \infty$

$\Rightarrow$ scale invariance, lattice $\rightarrow$ continuum, loops are fractal

(Conjecturally) Local conformal invariance: invariance under all maps that preserve angles³

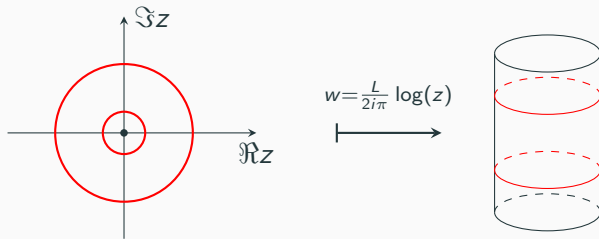
³Smirnov 2001, (percolation).

Critical limit

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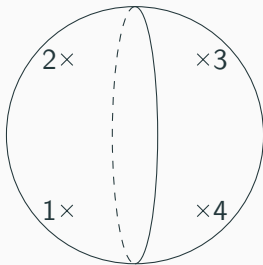
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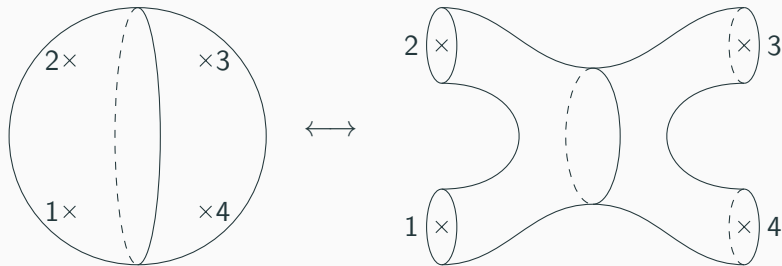


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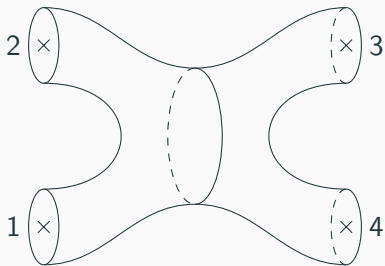
Complex surfaces



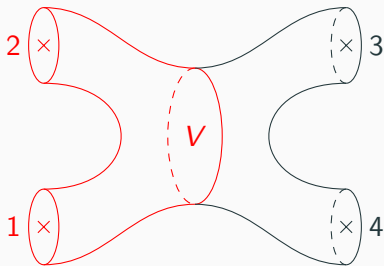
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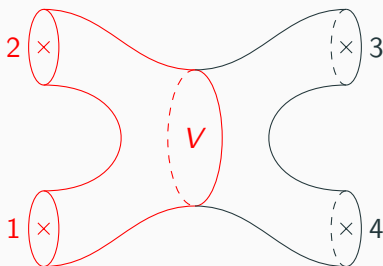
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Complex surfaces



Need to determine:

- Spectrum $\mathcal{S}$ (1987)
- Value of the pair of pants correlation function (2025⁴)
- How to glue pairs of pants (2026⁵)

⁴Jacobsen, Nivesvivat, Ribault, and Roux 2025.

⁵Roux, Jacobsen, and Saleur (to appear in 2026).

Spin and loop models

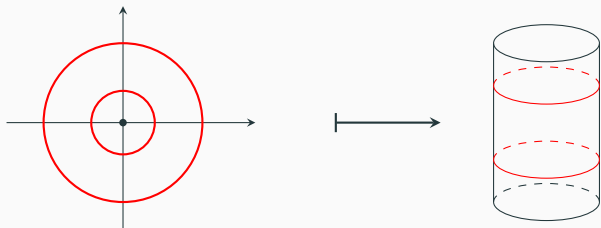
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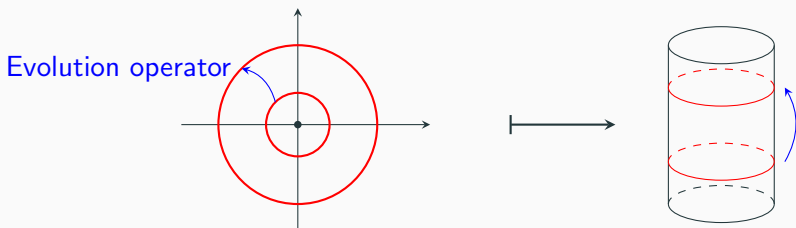
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Hamiltonian formalism



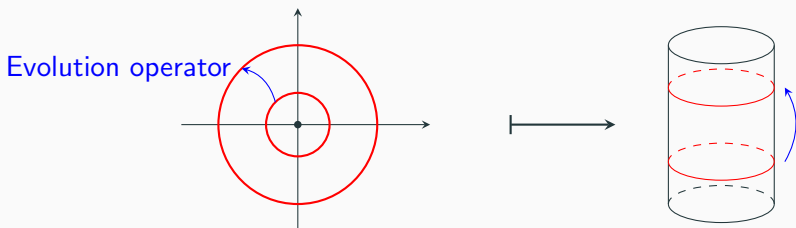
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Evolution operator: Hamiltonian H or transfer matrix T

Diagonalisation (integrability, numerics, or Coulomb gas):

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Diagonalisation (integrability, numerics, or Coulomb gas):

$$E_k - E_0 = \frac{2\pi(\Delta_k + \bar{\Delta}_k)}{L} + O\left(\frac{1}{L^2}\right)$$

$$\Delta - \bar{\Delta} = \text{spin}$$

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Result (1987⁶): for $V_{(r,s)}$, $(\Delta, \bar{\Delta}) = (\Delta_{(r,s)}, \Delta_{(r,-s)})$, where

$$\Delta_{(r,s)} = P_{(r,s)}^2 - P_{(1,1)}^2, \quad P_{(r,s)} = \frac{1}{2}(\beta r - \beta^{-1}s), \quad n = -2 \cos(\pi\beta^2)$$

⁶Di Francesco, Saleur, and Zuber 1987.

Bimodule decomposition

$$\mathcal{S} = \bigoplus_{(r,s) \in \frac{1}{2}\mathbb{N} \times \frac{1}{r}\mathbb{Z}} \mathcal{C}_{(r,s)} \otimes \Lambda_{(r,s)}$$

where $\mathcal{C}_{(r,s)}$ are modules of the conformal algebra of weights $(\Delta_{(r,s)}, \Delta_{(r,-s)})$, and^{7,8}

$$\Lambda_{(r,s)} = \bigoplus_{\substack{|\lambda| \leq 2r \\ |\lambda| \equiv 2r \pmod{2}}} c_{(r,s)}^\lambda \lambda ,$$

λ are irreducible representations of $O(n)$ (Deligne category)

⁷Jacobsen, Ribault, and Saleur 2023, (for $O(n)$, Potts).

⁸Roux, Jacobsen, Ribault, and Saleur 2025, (for $PSU(n)$).

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- $\Lambda_{(\frac{5}{2},\frac{2}{5})} = [41] \oplus [32] \oplus [31^2] \oplus [2^21] \oplus [21^3] \oplus [3] \oplus 2[21] \oplus [1^3] \oplus [1]$

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In general, $\Lambda_{(r,s)}$ not irreducible: symmetry $\supsetneq O(n)$

Spin and loop models

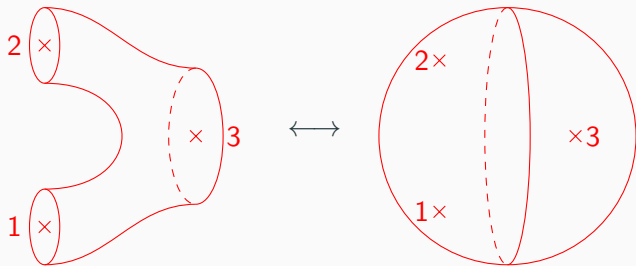
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Three-point functions

$$\left\langle \prod_{i=1}^3 V_{(r_i, s_i)}(x_i) \right\rangle = \frac{C_{(r_1, s_1)(r_2, s_2)(r_3, s_3)}}{\left| x_{12}^{\Delta_1 + \Delta_2 - \Delta_3} \right|^2 \left| x_{23}^{\Delta_2 + \Delta_3 - \Delta_1} \right|^2 \left| x_{13}^{\Delta_1 + \Delta_3 - \Delta_2} \right|^2}$$

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where $x_{ij} = x_i - x_j$, $|x^\Delta|^2 = x^\Delta \bar{x}^{\bar{\Delta}}$, $\Delta_i = \Delta_{(r_i, s_i)}$

$$|\mathcal{M}(r_1, r_2, r_3)| = \begin{cases} 1 & \text{if } r_1 + r_2 + r_3 \in \mathbb{N} \\ 0 & \text{else} \end{cases}$$

Three-point functions

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Conjecture:¹

$$C_{(r_1, s_1)(r_2, s_2)(r_3, s_3)} = \prod_{\epsilon_1, \epsilon_2, \epsilon_3 = \pm} \Gamma_{\beta}^{-1} \left(\frac{\beta + \beta^{-1}}{2} + \frac{\beta}{2} |\sum_i \epsilon_i r_i| + \frac{\beta^{-1}}{2} \sum_i \epsilon_i s_i \right)$$

¹Jacobsen, Nivesvivat, Ribault, and Roux 2025

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Particular cases:

- $r_i = 0$: structure constants of Liouville theory at $c \leq 1$ and minimal models
- $r_i = 1, s = 0$: three points on the same loop

$$\mathbb{P} \left(\begin{array}{c} \times \\ \diagup \quad \diagdown \\ \times \quad \times \end{array} \right) = C_{(1,0)(1,0)(1,0)}$$

Three-point functions

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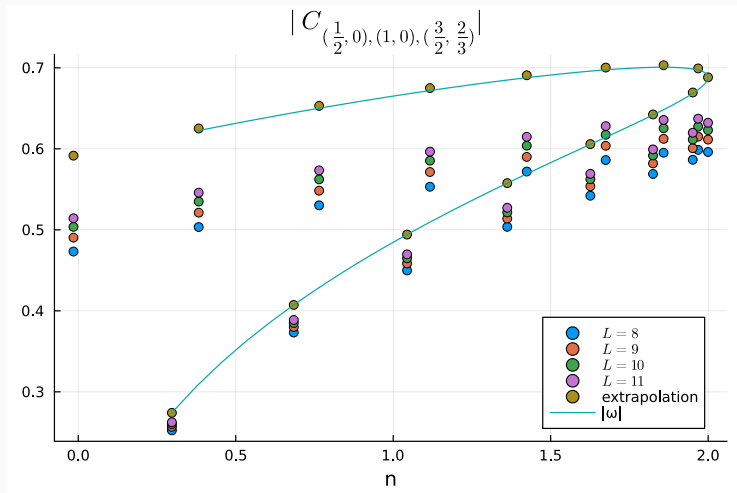
$$\mathbb{P} \left(\text{triangle diagram} \right) = C_{(1,0)(1,0)(1,0)}$$


Agrees with rigorous CLE result²

²Ang, Cai, Sun, and Wu 2024

Numerical evidence

On lattices of finite circumference L , extrapolated to $L \rightarrow \infty$:



+11 examples.

Spin and loop models

Spectrum

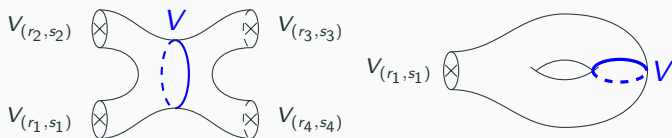
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Conformal block decomposition

Cutting of a surface into pairs of pants $\Leftrightarrow$ decomposition into conformal blocks



$$\left\langle \prod_{i=1}^N V_{(r_i, s_i)}(x_i) \right\rangle = \sum_{V \in \mathcal{S}} \underbrace{D_V}_{\text{structure constant}} \underbrace{\widehat{\mathcal{G}}_V}_{\text{conformal block}}(x_i)$$

Bootstrap equations

Bootstrap equations: correlation independent of decomposition:

$$\sum_V D_V^{(s)} \mathcal{G}_V^{(s)} = \sum_V D_V^{(t)} \mathcal{G}_V^{(t)}$$

Numerical bootstrap⁷:

$$D_V^{N=4, \text{sphere}} = \overbrace{C_{12V} C_{V34}}^{\text{geometric part}} \overbrace{g_V^{\langle V_1 V_2 V_3 V_4 \rangle \text{sphere}}}^{\text{tensorial part}}$$

$$D_V^{N=1, \text{torus}} = C_{1VV} g_V^{\langle V_1 \rangle \text{torus}}$$

Liouville, minimal models: $g_V = 1$

Goal: understand g_V

⁷Nivesvivat, Ribault, and Jacobsen 2024; Roux, Ribault, and Jacobsen 2026.

Numerical bootstrap results

$$g_{(r,s)}^{\langle V_{(1,0)} \rangle} = \frac{1}{\kappa_{(r,s)}} \left(\frac{d_{(r,s)}}{r} + \frac{\Theta_1^{r,s}}{n - w_{(r,s)}} \right) ,$$

$$w_{(r,s)} = 2 \cos 2\pi(r\beta^2 - s), \quad \kappa_{(\frac{1}{2},s)} = 2, \quad \kappa_{(1,s)} = 2, \quad \dots$$

$$\Theta_1^{r,s} = (-)^{r_1 s + s + 1} \prod_{j=\frac{1}{2} \frac{r_1 + 1}{2} - r}^{r - \frac{r_1 + 1}{2}} 2 \cos \pi \left(j\beta^2 - \frac{s_1}{2} \right) .$$

$$d_{(\frac{1}{2},0)} = 1, \quad d_{(1,0)} = n + 1, \quad d_{(1,1)} = n - 1$$

$$d_{(\frac{3}{2},0)} = (-n - 2)n^2 - n + 2$$

$$d_{(3,0)} = n^2 \left\{ (n^2 - 4)^2 n^5 - 4(n^4 - 9n^2 + 20)n^3 + (n^2 - 4)^2 n^2 \right. \\ \left. + (5n^4 - 40n^2 + 92)n + n^4 - 4n^3 + 16n - 4 \right\}$$

Algebraic derivation of g_V

$$\mathcal{S} = \bigoplus_{(r,s) \in \frac{1}{2}\mathbb{N} \times \frac{1}{r}\mathbb{Z}} \mathcal{C}_{(r,s)} \otimes \Lambda_{(r,s)}$$

$P_{(r,s)}$ = projectors onto $\Lambda_{(r,s)}$, i.e. simple idempotents in the Temperley–Lieb category, $\pi_{(r,s)}$ projectors onto $\mathcal{C}_{(r,s)}$:

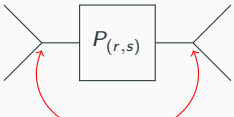
$$\text{Id}_{\mathcal{S}} = \sum_{(r,s)} \pi_{(r,s)} \otimes P_{(r,s)}$$

Example:

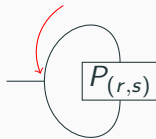
$$P_{(1,0)} = \frac{1}{2} \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} \right) - \frac{1}{4n} \left(\begin{array}{c} \bullet \quad \bullet \\ \cup \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \cap \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \cup \\ \bullet \quad \bullet \end{array} \right)$$

Extended global symmetry

Computation of g_V :

$$g_{V_{(r,s)}}^{\langle V_1 V_2 V_3 V_4 \rangle} =$$


unique way to connect the legs

$$g_{V_{(r,s)}}^{\langle V_1 \rangle} =$$


Extended global symmetry

Computation of g_V :

$$g_{V_{(r,s)}}^{\langle V_1 \rangle} = \text{---} \bigcirc \boxed{P_{(r,s)}}$$

Example: $V_1 = V_{(1,0)} = V$

$$g_{V_{(1,0)}}^{\langle V_{(1,0)} \rangle} = \text{---} \bigcirc \boxed{P_{(1,0)}}$$

Extended global symmetry

Computation of g_V :

Example: $V_1 = V_{(1,0)} = V$

$$g_{V_{(1,0)}}^{\langle V_{(1,0)} \rangle} = \text{diagram} = \frac{1}{2} \left(\text{diagram}_1 + \text{diagram}_2 \right) - \frac{1}{4n} \left(\text{diagram}_3 + \text{diagram}_4 + \text{diagram}_5 + \text{diagram}_6 \right)$$

The diagrams are as follows:

- Initial Diagram:** A genus-1 surface (torus) with a box labeled $P_{(1,0)}$ on its right side. The box has four vertices, each with a dot. A horizontal line with a dot enters from the left and connects to the top-left vertex of the box. Two loops are shown: one encircling the handle and another encircling the box.
- Diagram 1 (in the $\frac{1}{2}$ term):** A genus-1 surface with a box containing two vertical lines, each with two dots. A horizontal line with a dot enters from the left and connects to the top-left dot.
- Diagram 2 (in the $\frac{1}{2}$ term):** A genus-1 surface with a box containing two vertical lines, each with two dots. A horizontal line with a dot enters from the left and connects to the top-left dot. There are additional internal connections within the box.
- Diagram 3 (in the $-\frac{1}{4n}$ term):** A genus-1 surface with a box containing two vertical lines, each with two dots. A horizontal line with a dot enters from the left and connects to the top-left dot. Internal connections are different from Diagram 1.
- Diagram 4 (in the $-\frac{1}{4n}$ term):** A genus-1 surface with a box containing two vertical lines, each with two dots. A horizontal line with a dot enters from the left and connects to the top-left dot. Internal connections are different from Diagram 2.
- Diagram 5 (in the $-\frac{1}{4n}$ term):** A genus-1 surface with a box containing two vertical lines, each with two dots. A horizontal line with a dot enters from the left and connects to the top-left dot. Internal connections are different from Diagram 3.
- Diagram 6 (in the $-\frac{1}{4n}$ term):** A genus-1 surface with a box containing two vertical lines, each with two dots. A horizontal line with a dot enters from the left and connects to the top-left dot. Internal connections are different from Diagram 4.

Extended global symmetry

Computation of g_V :

Example: $V_1 = V_{(1,0)} = V$

$$\begin{aligned}
 g_{V_{(1,0)}}^{\langle V_{(1,0)} \rangle} &= \text{diagram of a torus with a box labeled } P_{(1,0)} \text{ and four vertices} = \frac{1}{2} \left(\text{diagram 1} + \text{diagram 2} \right) \\
 &\quad - \frac{1}{4n} \left(\text{diagram 3} + \text{diagram 4} + \text{diagram 5} + \text{diagram 6} \right) \\
 &= \frac{1}{2} \left(n + 1 - \frac{1}{n} \right) \quad \text{in agreement with bootstrap results}
 \end{aligned}$$

Spin and loop models

Spectrum

Sphere 3 point functions

Sphere four-point and torus one-point functions

Conclusion

General conjecture

In 2D, symmetry of the $O(n)$ model $\equiv$ Temperley-Lieb category.

Conjecture: structure constants g_V are given by fusion diagrams:

- Simple objects: $P_{(r,s)}$
- Junction spaces $N_{(r_1,s_1)(r_2,s_2)}^{(r_3,s_3)} = |\mathcal{M}(r_1, r_2, r_3)| \in \{0, 1\}$
- F -symbols:

$$(F_{V_4}^{V_1 V_2 V_3})_{V'}^{V'} = \text{Diagram: A circle divided into four sectors by two lines meeting at a central point. The top-left sector is labeled } V_1, \text{ the top-right is } V_2, \text{ the bottom-left is } V, \text{ and the bottom-right is } V_3. \text{ The label } V_4 \text{ is outside the circle at the top-left, and } V' \text{ is outside at the top-right.}$$

$\Rightarrow$ general expressions for correlation functions: channel decomposition $\leftrightarrow$ fusion diagram, e.g.

$$\langle V_1 V_2 V_3 V_4 \rangle^{\text{sphere}} = \sum_{V \in \mathcal{S}} \text{Diagram: A fusion diagram with four external legs labeled } V_1, V_2, V_3, V_4 \text{ and a central horizontal line. } V_1 \text{ and } V_2 \text{ are on the left, } V_3 \text{ and } V_4 \text{ are on the right.} C_{V_1 V_2 V} C_{V V_3 V_4} g_V$$

General conjecture

In 2D, symmetry of the $O(n)$ model $\equiv$ Temperley-Lieb category.

Conjecture: structure constants g_V are given by fusion diagrams:

- Simple objects: $P_{(r,s)}$
- Junction spaces $N_{(r_1,s_1)(r_2,s_2)}^{(r_3,s_3)} = |\mathcal{M}(r_1, r_2, r_3)| \in \{0, 1\}$
- F -symbols:

$$(F_{V_4}^{V_1 V_2 V_3})_{V'}^{V'} = \text{Diagram}$$

$\Rightarrow$ general expressions for correlation functions: channel decomposition $\leftrightarrow$ fusion diagram, e.g.

$$\langle V_1 \rangle^{\text{torus}} = \sum_{V \in \mathcal{S}} v_1 \text{Diagram} C_{V_1 V V} g_V$$

General conjecture

In 2D, symmetry of the $O(n)$ model $\equiv$ Temperley-Lieb category.

Conjecture: structure constants g_V are given by fusion diagrams:

- Simple objects: $P_{(r,s)}$
- Junction spaces $N_{(r_1,s_1)(r_2,s_2)}^{(r_3,s_3)} = |\mathcal{M}(r_1, r_2, r_3)| \in \{0, 1\}$
- F -symbols:

$$(F_{V_4}^{V_1 V_2 V_3})_{V'}^{V''} = \text{fusion diagram}$$

$\Rightarrow$ general expressions for correlation functions: channel decomposition $\leftrightarrow$ fusion diagram, e.g.

$$Z(\text{genus} = 2, x) = \sum_{V_1, V_2, V_3} \text{fusion diagram} \mathcal{G}_{V_1, V_2, V_3}(x)$$

- OPEs are not valid in a naive sense:

$$V_1(x_1)V_2(x_2) = \sum_k C_{V_1 V_2}^{V_k} V_k(x_2),$$

$$C_{V_1 V_2}^{V_k} = C_{V_1 V_2 V_k} \times (\text{morphism: } \Lambda_{(r_1, s_1)} \times \Lambda_{(r_2, s_2)} \rightarrow \Lambda_{(r_k, s_k)})$$

- limits $n \rightarrow 0, 1, \dots$: correlations of percolation, polymers, ...
- Extended symmetry $\supsetneq O(n)$ specific to 2D: non-crossing of loops.
 - relation with the $O(n)$ model in $D \neq 2$?
 - 2D CFT with $O(n)$ global symmetry?

