Conformal invariance of the Ising model

Les Gustins

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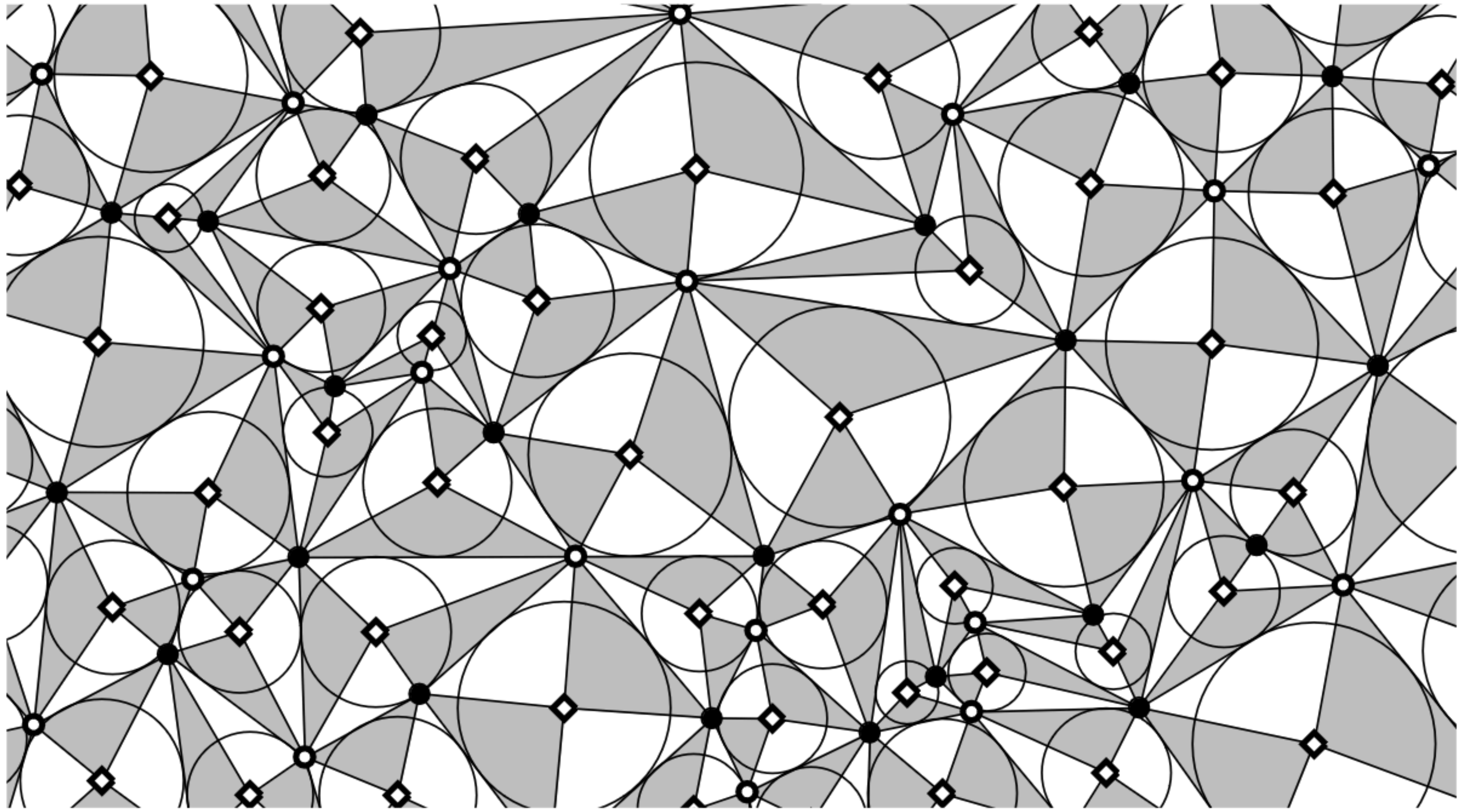


Classical model on the square lattice

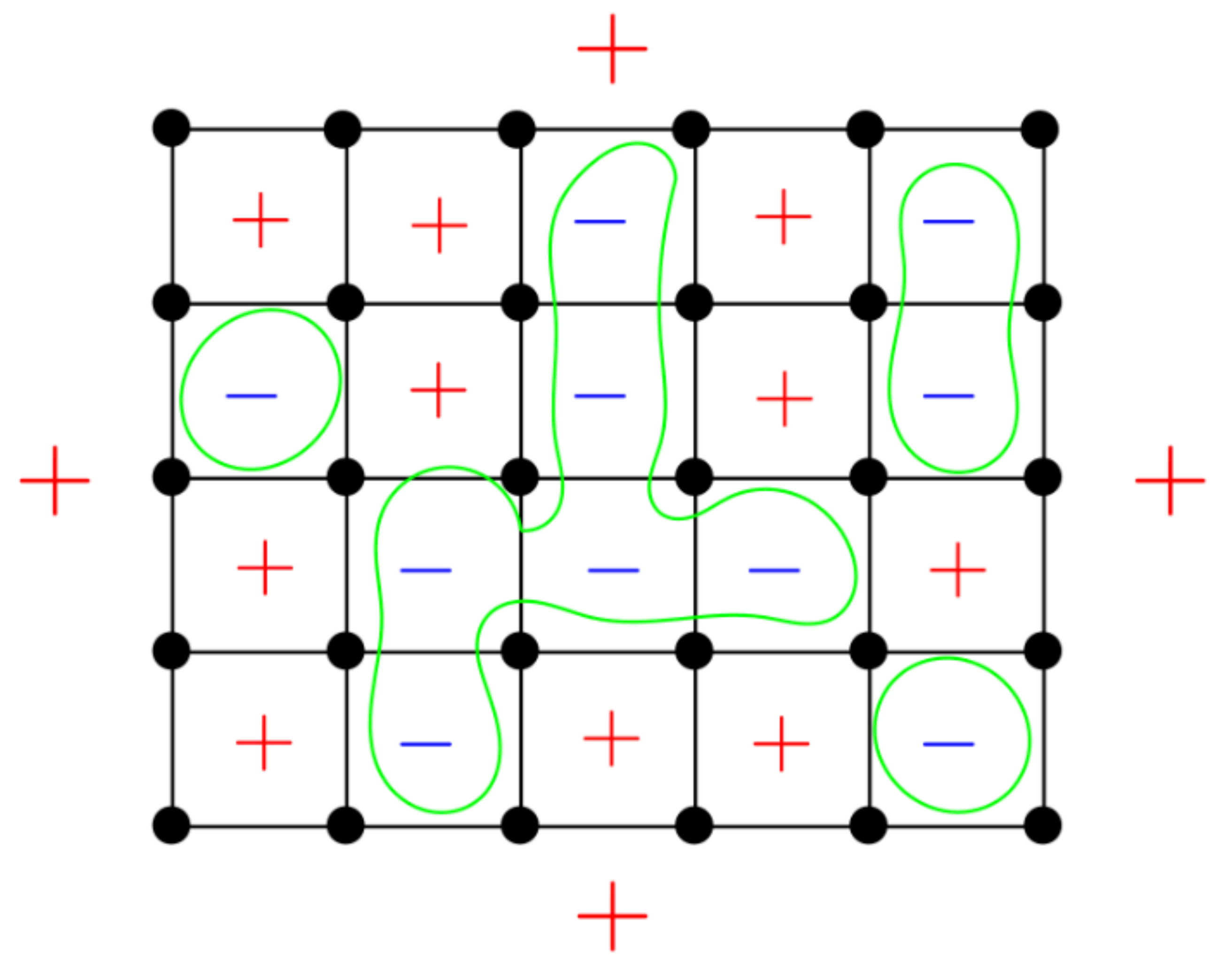
- $\mathscr{G}$ finite subgraph of $\mathbb{Z}^2$
- $\mathscr{F}$ the set of faces $\mathscr{G}$
- Fix $x \in]0;1[$, represents the cost of non-aligned neighbours

Spin configuration

$\sigma = (\sigma_u)_{u \in \mathscr{F}} \in \{\pm 1\}^{\mathscr{F}}$

+1 -1

Collection of contours separating opposite spins



$$\mathbb{P}^+_{\mathscr{G}}(\sigma) := \frac{1}{Z(\mathscr{G}, x, +)} x^{\#(uv)\in\mathscr{G}, \sigma_v \neq \sigma_v}$$

Partition function

$$Z(\mathscr{G}, x, +) := \sum_{\sigma\{\pm 1\}^{\mathscr{F}}} x^{\#(uv)\in\mathscr{G}, \sigma_v \neq \sigma_v}$$

Full plane measures and phase transition

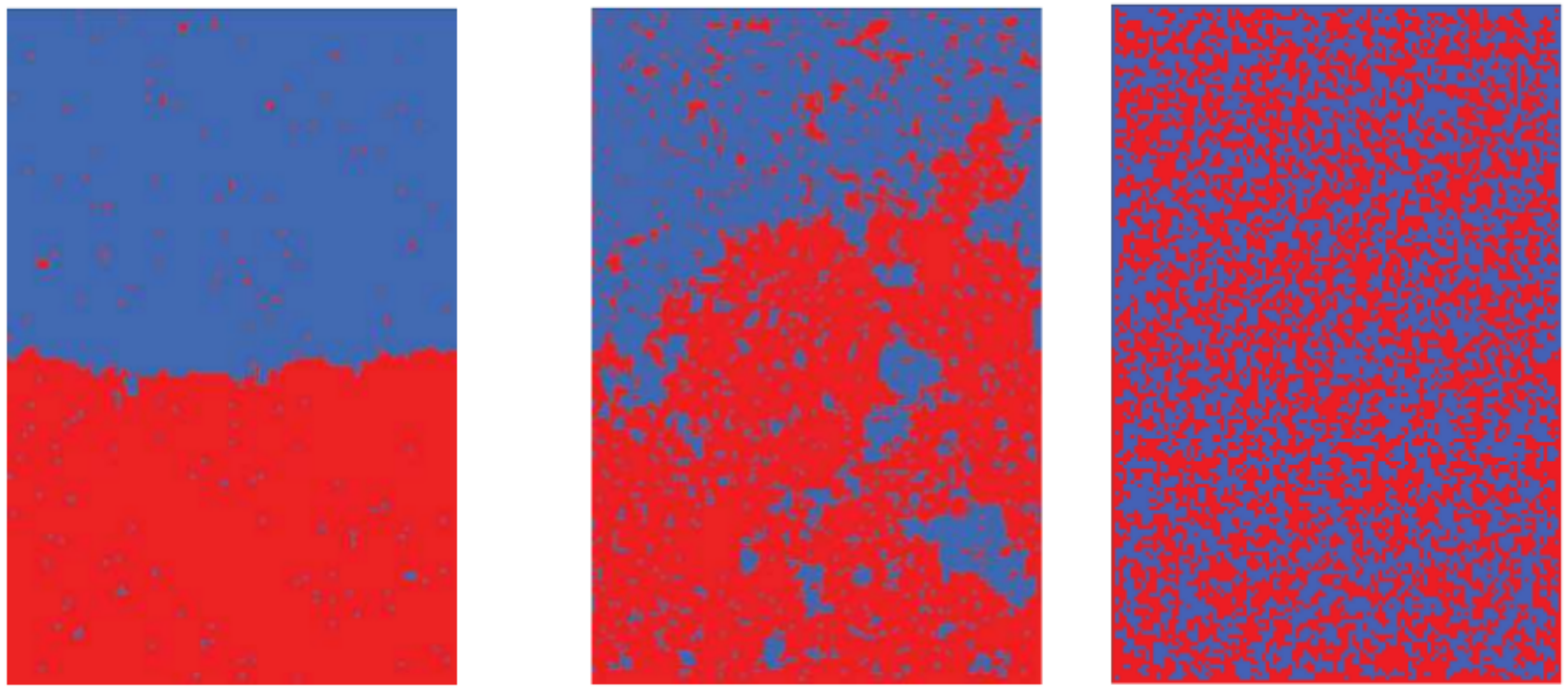
- Monotonicity corresponding to the physical intuition. Full plane (a priori different) Gibbs measures allows to construct full plane on $\mathbb{Z}^2$

$$\mathbb{E}^{+}_{\mathcal{G}_n}[\sigma_0] \geq \mathbb{E}^{+}_{\mathcal{G}_{n+1}}[\sigma_0]$$

- Second order phase transition

<u>Free energy per site</u>

$$\boxed{f(x) := \lim_{n\infty} \frac{1}{|\mathcal{G}_n|} \log Z(\mathcal{G}_n, x, +)}$$



$x > x_c$ $\quad$ $x_c = \sqrt{2} - 1$ $\quad$ $x < x_c$

- Non trivial critical point Peierls (1936)
- Value of critical point by Onsager (1944)
- Many exact computation using integrability techniques (McCoy, Wu, Baxter, Perk …)

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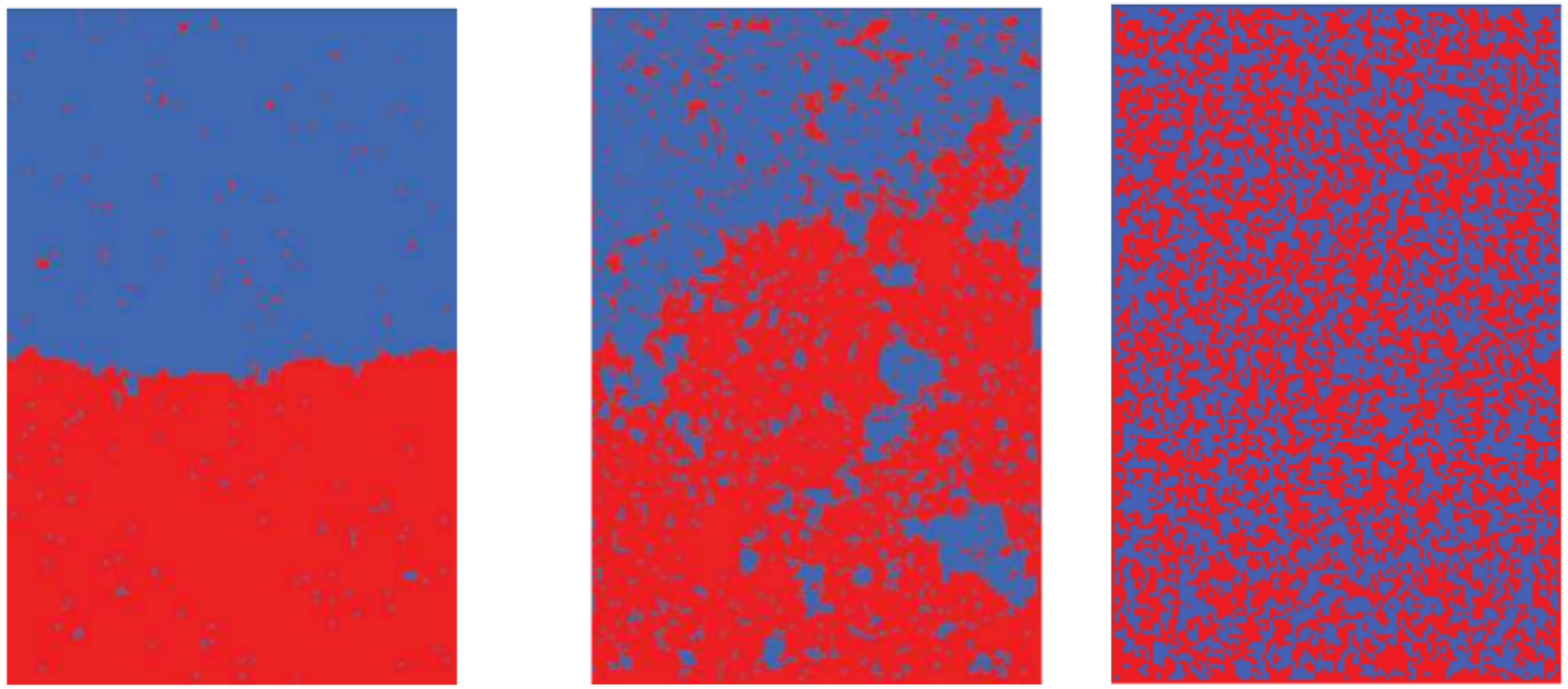


$x > x_c$ $\quad$ $x_c = \sqrt{2} - 1$ $\quad$ $x < x_c$

$$\mathbb{E}^+[\sigma_u \sigma_v] \sim (1 - \sinh^4(2J))^{\frac{1}{4}}$$

$$\mathbb{E}^+[\sigma_u \sigma_v] \sim \exp(-c(x)|u - v|)$$

$$\mathbb{E}^+[\sigma_u \sigma_v] \sim \frac{1}{|y - v|^{\frac{1}{4}}}$$

<u>Free energy per site</u>

$$f(x) := \lim_{n\infty} \frac{1}{|\mathcal{G}_n|} \log Z(\mathcal{G}_n, x, +)$$

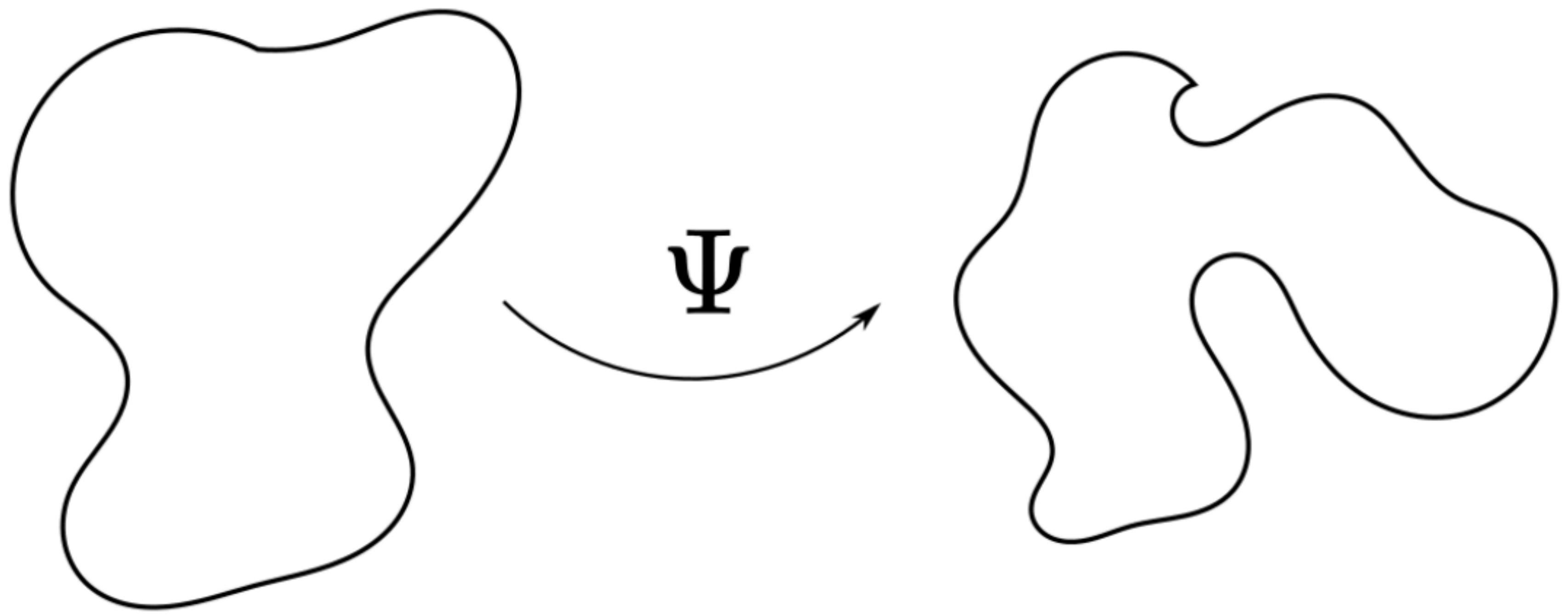
$$x = e^{-2J}$$

Conformally invariant scaling limit at criticality

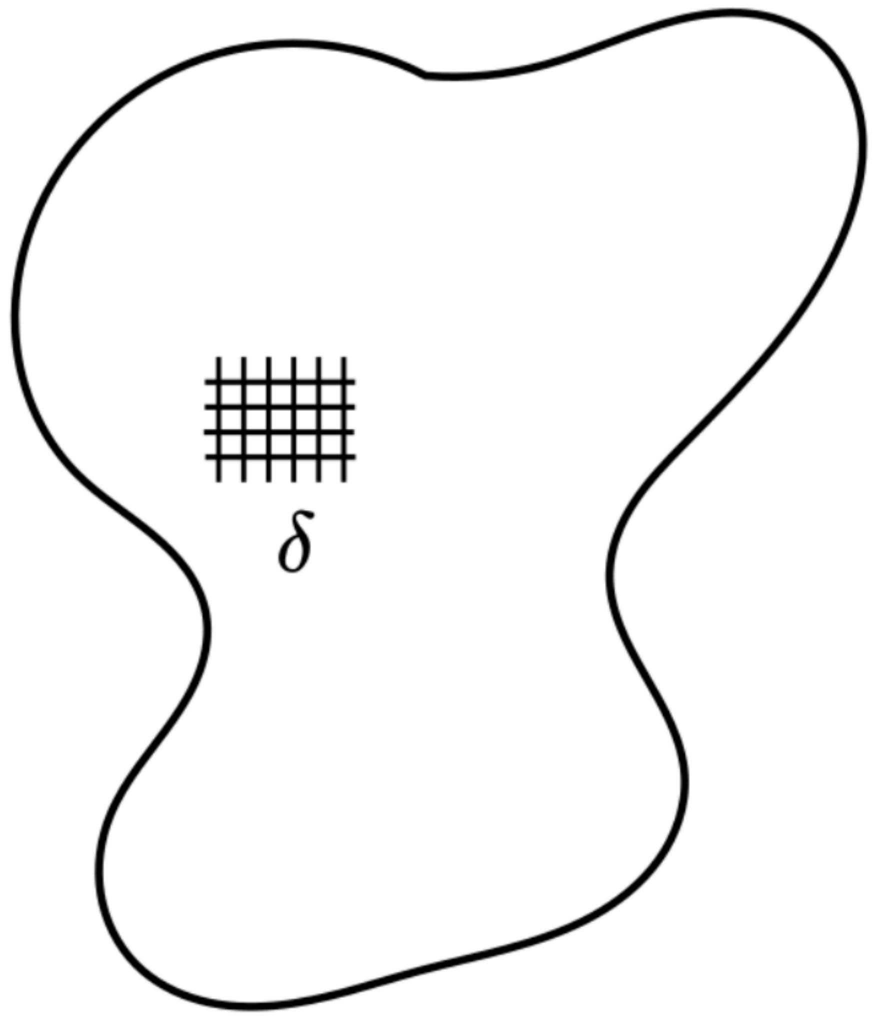
- Predicted by CFT physicists in the 1980's : Belavin, Polyakov, Zamolodchikov
- Numerical simulations made by Langland and coauthors
- If there is a scaling limit

- Introduction of SLE's by Schramm
- Introduction of discrete complex

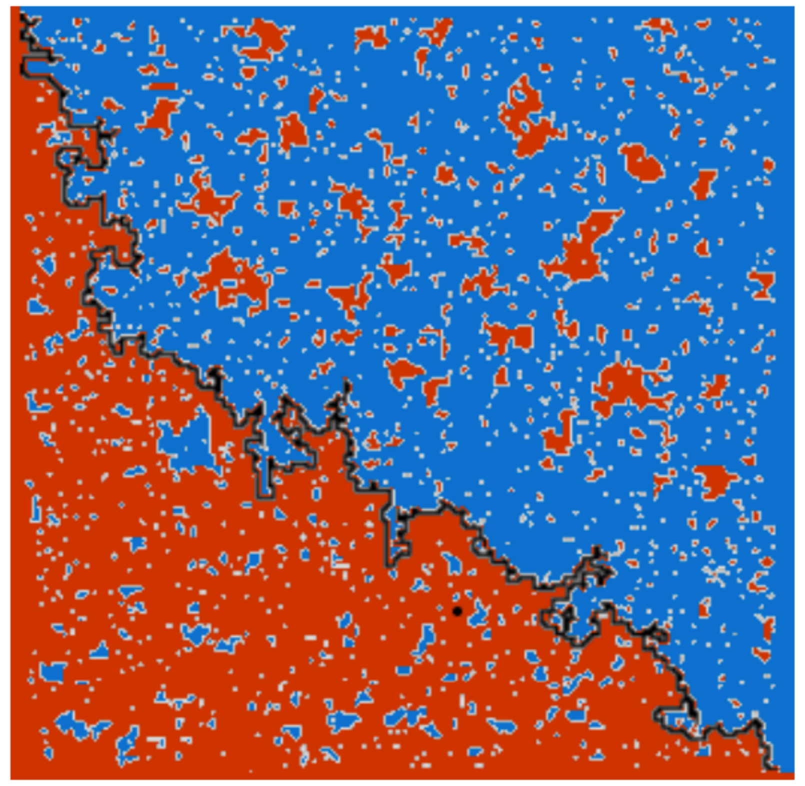
Analysis techniques
Observables by Kenyon (dimers) and Smirnov (Percolation)



$\Psi($ scaling limit $)=$ scaling limit $\Psi(\Omega)$

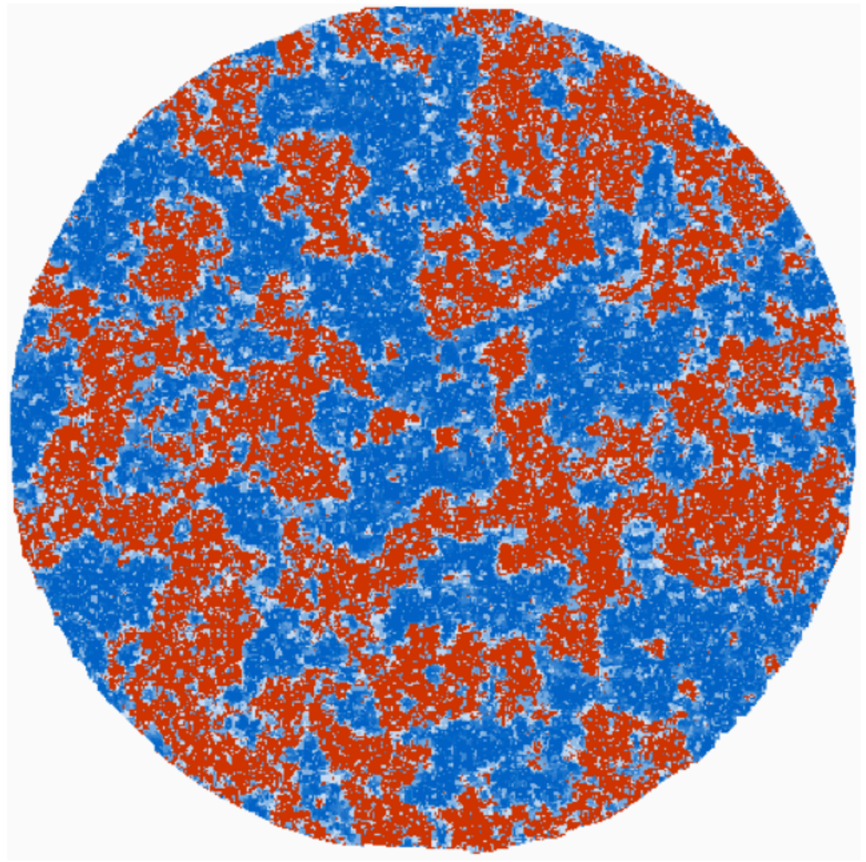


Conformally invariant scaling limit at criticality



$$\Psi\left[\mathrm{SLE}(\Omega)\right] = \mathrm{SLE}\left(\Psi\left[\Omega\right]\right)$$

- Convergence of FK-Interfaces to SLE(16/3) by Smirnov (2006)
- Convergence of the loop ensemble to CLE(3) by Hongler and Benoist

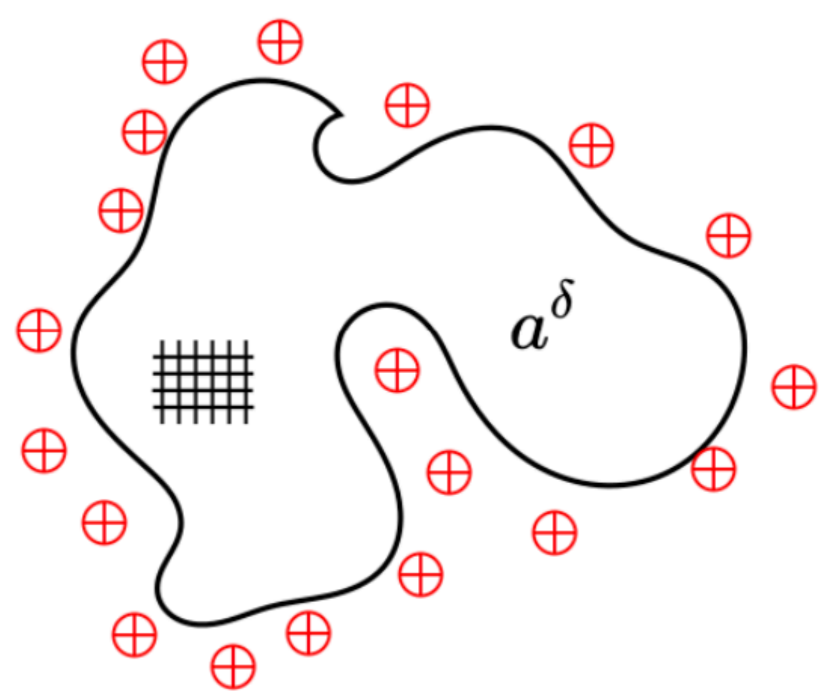


$$\Psi\left[\mathrm{CLE}(\Omega)\right] = \mathrm{CLE}\left(\Psi\left[\Omega\right]\right)$$

Conformally invariant scaling limit at criticality

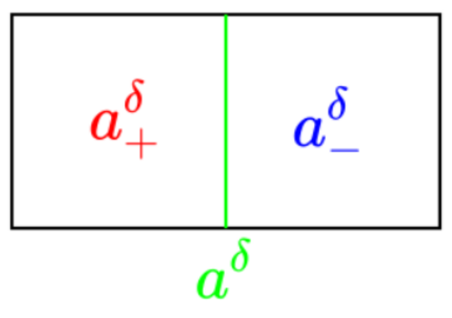
- Convergence of the energy field by Hongler and Smirnov (2013)

$$\sqrt{2}\frac{\mathbb{E}^+_{\Omega^\delta}[\sigma_{a^+}\sigma_{a^-}] - \mathbb{E}^+_{\mathbb{C}^\delta}[\sigma_{a^+}\sigma_{a^-}]}{\delta} \to \ell_\Omega(a)$$

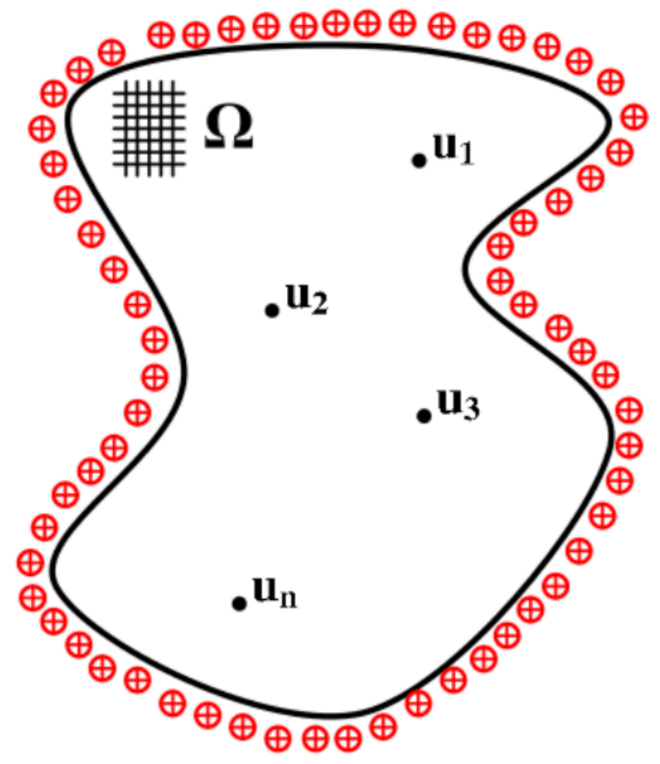


$$\ell_\Omega(a) = |\Psi'(a)|\,\ell_{\Psi(\Omega)}(\Psi(a))$$

- Convergence of general spin correlation (Chelkak, Hongler and Izyurov)



$$\delta^{-\frac{n}{8}}\mathbb{E}^+_{\Omega^\delta}[\sigma_{u_1^\delta}\cdots\sigma_{u_n^\delta}] \to \langle \sigma_{u_1}\cdots\sigma_{u_n}\rangle^+_\Omega$$



Fusion rules by Chelkak Hongler Izyurov

$$\langle \sigma_{u_1}\cdots\sigma_{u_n}\rangle^+_\Omega = \langle \sigma_{\Psi(u_1)}\cdots\sigma_{\Psi(u_n)}\rangle^+_{\Psi(\Omega)}\prod_{i\le n}|\Psi'(u_i)|^{\frac{1}{8}}$$