DATUM/DATE 23/08/23 Ludwig Hruza

Presentation: Time in QM and the precision of quantum clocks

(I) Time - Energy - Uncertainty

1) QM Textbooks (Sakurai)

- $|\psi(0)\rangle = \sum_n c_n |E_n\rangle$ initial state with mean energy $E = \sum_n |c_n|^2 E_n$ and width $(\Delta E)^2 = \langle (H-E)^2 \rangle = \sum_n |c_n|^2 (E_n - E)^2$ will evolve to a considerably different state $|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |E_n\rangle$ with $\langle \psi(t)|\psi(0)\rangle \ll 1$ in a time period $\Delta t > \frac{\hbar}{\Delta E}$

Derivation

$$\langle \psi(t)|\psi(0)\rangle = e^{-iEt/\hbar} \sum_n |c_n|^2 e^{-i(E_n - E)t/\hbar} \ll 1$$

as soon as $e^{-i(E_n - E_0)t/\hbar}$ oscillades rapidly with n and cancels. This happens for $t = \alpha \hbar/\Delta E$, $\alpha > 1$ since only those terms contribute where $E_n - E \lesssim \Delta E$

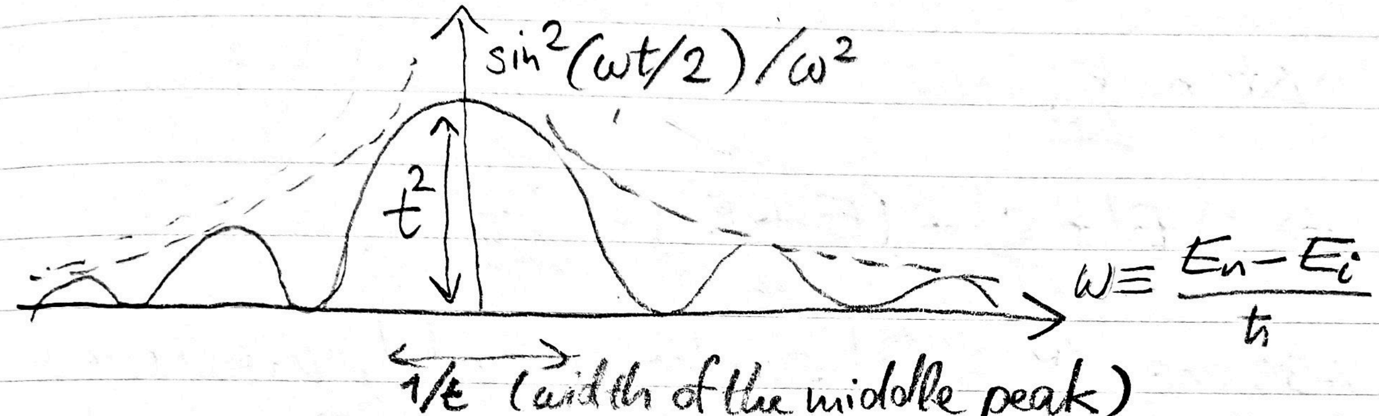
- Time dep. perturbation theory : $V(t) = \Theta(t)\, V$.

Prob ($|i\rangle \longrightarrow |n\rangle$ at time t)

1st order $= \dfrac{4|V_{in}|^2}{|E_n - E_i|^2} \sin^2\left(t\, \dfrac{E_n - E_i}{2\hbar}\right)$

$\neq 0$ if $t \sim \dfrac{\hbar}{|E_n - E_i|} \equiv \dfrac{\hbar}{\Delta E}$

Transition prob from initial state $|i\rangle$ with energy E_i to $|n\rangle$ with energy E_n



2) "Common Interpretation" of $\Delta E\, \Delta t \sim \hbar$?

where Δt : time of measurement of energy

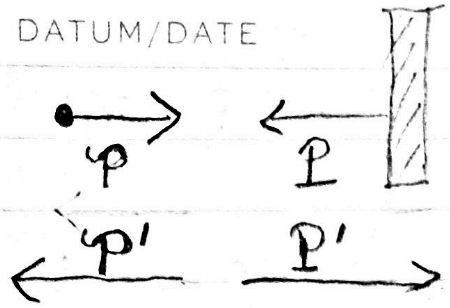
ΔE : Uncertainty of the energy measurement

or

Error introduced to the system due to the measurement

- Landau & Peierls (1931)

Measure a free particles energy $E = \dfrac{P^2}{2m}$ by a momentum measurement with a test particle (a mirror).



$$p' + P' = p + P$$

$$E' + \varepsilon' - (E + \varepsilon) = 0$$

(Elastic collision)

Collision via a potential V during Δt

Time dep. perturb. theory:
Transition to another Energy state of the combined system is possible if

$$\Delta t \sim \frac{\hbar}{\Delta E}$$

$$\Rightarrow |E' + \varepsilon' - (E + \varepsilon)| \geq \frac{\hbar}{\Delta t}$$

Assume P & P' can be perfectly measured ($\Delta P = 0 = \Delta P'$) $\Rightarrow \Delta E = 0 = \Delta E'$
then

$\Delta(\varepsilon' - \varepsilon) \geq \frac{\hbar}{\Delta t}$, i.e. the measurement during Δt transfers a bit of energy to the system / introduces an uncertainty

From $\varepsilon = p^2/2m$ we have

$$\Delta(\varepsilon' - \varepsilon) = \frac{p' - p}{m} \Delta p \geq \frac{\hbar}{\Delta t}$$

i.e. the measurement introduces an error in the particle's momentum of

$$\Delta p \geq \frac{\hbar}{(v' - v)\Delta t} \quad \text{with } v = \frac{p}{m}$$

- Objection by Bohm & Aharonov (1961)

choosing a suitible interaction, energy (via the momentum) can be measured to any accuracy in any short time period Δt

$$H = \underbrace{\frac{p^2}{2m}}_{\text{observed Particle}} + \underbrace{\frac{P^2}{2M}}_{\text{test Particle}} + X p \underbrace{g(t)}_{g\,\mathbb{1}(t_0<t<t_0+\Delta t)}$$

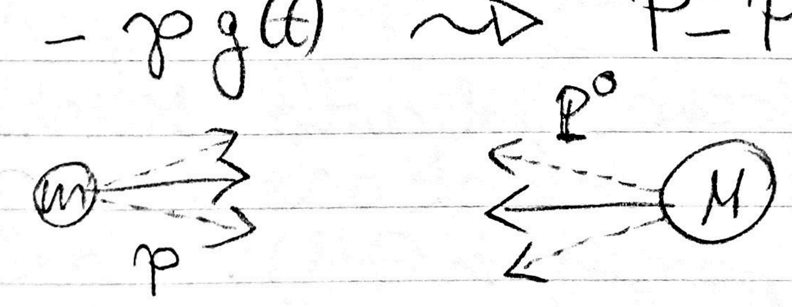
$$\dot{x} = i[H, x] = \frac{p}{m} + X g(t)$$

$\dot{p} = 0$ (momentum is not affected since V_{int} is indep of x)

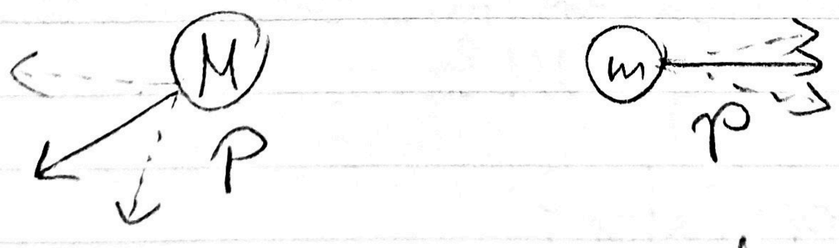
$$\dot{X} = \frac{P}{M}$$

$$\dot{P} = -p\,g(t) \rightsquigarrow P - P^0 = -p\,g\,\Delta t$$

before



after



To obtain p up to Δp we need ΔP^0 to be small enough s.t. $\Delta(P - P^0) = \Delta p\, g\, \Delta t \geq \Delta P^0$

i.e. choosing g large, $\Delta p \Delta t\, g \geq \Delta p_0$ is satisfied for any Δp and Δt, small as one wishes.
(small $\Delta p \rightsquigarrow$ small $\Delta E = \frac{p \Delta p}{m}$)

Experimental implementation via two condensators that give pulses at t and $t + \Delta t$ st. $\dot{x}$ changes accordingly without affecting p.

Take away: The "common interpretation" of $\Delta E \Delta t \sim \hbar$ with Δt measured by some external observer is not always satisfied and dep. of the specific measurement protocol.

② Quantum Clocks

Until now: concerned with "external time" measured by a clock outside the system (time is a param. in QM). Could we include the clock into the system and describe it by QM?

1) A time operator?

• If we had one, T, st. $\frac{d}{dt} T = \mathbb{1}$, then we could get a standard uncertainty relation between conjugate Observables

$\Delta A := A - \langle A \rangle$

$\Delta B := B - \langle B \rangle$ some operators

Schwarz inequality $(\langle \alpha | \alpha \rangle + \langle \beta | \beta \rangle \geq |\langle \alpha | \beta \rangle|^2)$ implies

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |\langle [A, B] \rangle|^2$$

- Pauli (1933): time operators are unphysical

Assume T exists st. $\partial_t T = i[H, T] = \mathbb{1}$

then $e^{i\lambda T} H e^{-i\lambda T} = H + i\lambda [T, H] + 0$
$= H - \lambda \mathbb{1}$ for $\lambda \in \mathbb{R}$

and (spectrum of H) $= \mathbb{R}$.

$\Rightarrow$ Unbounded spectrum of H is unphysical! I.e. an ideal quantum clock is not possible

(also: H cannot have a cont. spectrum in a finite sys.)

- Wigner & Salecker (1957)

Approximate an ideal clock by a finite dim. quantum system and find its accuracy as a function of its mass.

- **Peres (1980)** (based on Wigner's clock)

Hilbertspace dimension $d = 2j+1$

Basis (in Energy): $\{|E_n\rangle\}_{n=-j}^{j}$

Clock Hamiltonian: $H_c = \sum_{n=-j}^{j} \hbar n\omega |E_n\rangle\langle E_n|$

(harmonic oscillator with energy quanta $\hbar\omega$ and d states)

Complementary basis (time basis)

$$|t_k\rangle = \frac{1}{\sqrt{d}} \sum_n e^{-2\pi i k n/d} |E_n\rangle$$

"Time operator"

$$T_c = \tau \sum_{k=0}^{d-1} k P_k$$ where $P_k = |t_k\rangle\langle t_k|$ projector on $|t_k\rangle$

τ: time resolution (smallest time interval) of the clock

$$\tau := \frac{2\pi}{\omega \cdot d} = \frac{\text{Period of the oscillator}}{\text{\#states it runs through}}$$

$$e^{-iH_c\tau/\hbar} |t_k\rangle = \frac{1}{\sqrt{d}} \sum_n e^{-2\pi i k n/d} e^{-in\omega\tau} |E_n\rangle$$

$$= |t_{k+1}\rangle$$

⇝ Clock Hamiltonian moves $|t_k\rangle \to |t_{k+1}\rangle$ in a timestep τ.

$$\boxed{\begin{array}{l} \text{Explicit realisation of } \mathcal{H} = L^2[0,2\pi) \\ \theta \in [0,2\pi) \\ \langle \theta | E_n \rangle = \frac{1}{\sqrt{2\pi}} e^{in\theta} \qquad n = -j, \dots, j \\ H_c = \omega\,(-i\hbar)\partial_\theta = \omega J \end{array}}$$

Energy-uncertainty of "time basis"

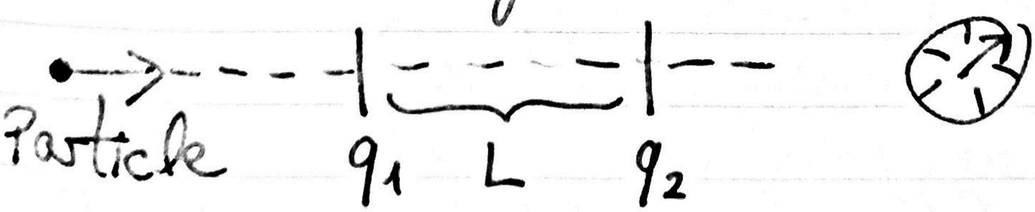
$$\langle t_k | H_c | t_k \rangle = \frac{1}{d} \sum_{n=-j}^{j} \hbar\omega n e^0 = 0$$

$$\langle t_k | H_c^2 | t_k \rangle = \frac{\hbar^2\omega^2}{d} \sum_{n=-j}^{j} n^2 = \frac{\hbar^2\omega^2}{3} j(j+1)$$

$$\rightsquigarrow \Delta H_c \simeq \frac{j\hbar\omega}{\sqrt{3}} \simeq \frac{\pi}{\sqrt{3}} \frac{\hbar}{\tau}$$

(the clock is a non-classical object)

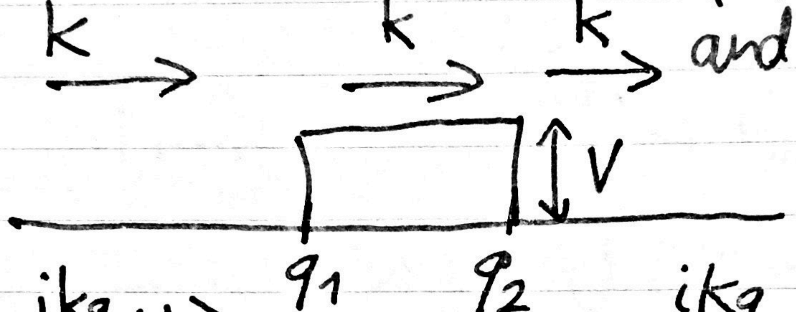
- <u>Time of flight measurement</u>



$$H = \frac{p^2}{2m} + P(q) \otimes H_c \quad \text{where } P(q) = \mathbb{1}(q_1 < q < q_2)$$

Initial clock state $|t_0\rangle = \frac{1}{\sqrt{d}} \sum_n |E_n\rangle$

$$\langle E_n | H | E_m \rangle = \frac{p^2}{2m} + \underbrace{P(q)\,\hbar\omega n}_{\text{Potential well of hight } V = n\hbar\omega \text{ and Length } L} \delta_{n,m}$$



$e^{ikq}|t_0\rangle$ $\qquad$ $e^{ikq} \sum_n e^{i\phi} |E_n\rangle$

$$\phi = (k' - k)L \approx -\frac{n\omega L}{\sqrt{2E/m}} = -n\omega T$$

time of flight

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$k' = \frac{\sqrt{2m(E-V)}}{\hbar} \approx \frac{\sqrt{2mE}}{\hbar} - \frac{V}{2\hbar}\sqrt{\frac{2m}{E}}$$

$|V| \ll E$

final state: $e^{ikq} \sum_n e^{-in\omega T} |E_n\rangle$

$$= e^{ikq} |t_{\frac{T}{\tau}}\rangle$$

<u>Accuracy</u> of the clock: We required $|V| \ll E$ st. the free particle is not much disturbed

$\max|V| \simeq j\hbar\omega \simeq \frac{\pi\hbar}{\tau}$. Hence, the clocks resolution is limited by $\tau \gg \frac{\hbar}{E}$.

- Use a clock not as a stopwatch (as before, passive, does not generate space-time events) but as a chimming clock that can <u>control a process</u>.

Here: Turn on a magn. field at time t_a, this makes a spin precess, turn it off at time t_b, measure the angle run through by the spin

$$H = H_C + (P_a + P_{a+1} + \ldots + P_{b-1})\, H_S$$

P_a: Projector on $|t_a\rangle$ giving a clocktime t_a

H_S: $\hbar\Omega\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, Ω: spin precession freq.

Once $|t_k\rangle$ is outside the range $a \leq k < b$ (due to the clock's evolution with H_C) the sum of projectors will project the current state to zero and the spin precession via H_S stops.

<u>Result</u>: the coupling clock-spin not only perturbs the spin, but in this case, also perturbs the clock itself (because H_C and P_k don't commute). $\tau \ll \frac{1}{\Omega}$, i.e. ~~the clocks resolution is lim~~ in order to limit the clock's perturbance, we can only controle a spin precession whose frequency is much ~~sm~~ smaller than the clock's frequency.

- Peres concludes: "results are self-defeating"
  - improving time resolution τ perturbes the system (time-of-flight-measurement)
  - using a clock to control another system, the clock itself is perturbed
  - ⇒ Peres questions the operational meaning of $\dot{\psi} = \lim \frac{\psi(t+\varepsilon) - \psi(t)}{\varepsilon}$ in the Schröd. equation since in his examples $\varepsilon > \tau$ is lowerbounded

- Problem of Peres' clock: No mechanism to output time. If we measure the clock with a projective measurement in the basis $\{|t_k\rangle\}$, at the wrong moment, eg. at $t = (k+\frac{1}{2})\tau$ then
  $\langle t_{k+1/2} | t_n \rangle \neq 0 \quad \forall n = 0, \dots, d-1$
  and the clock is irreversibly perturbed

- Solution: "Quasi-ideal clock" (Oppenheim, Woods, Silva 2018): Use a Gaussian envelope for the initial clock state

$$|\psi(k)\rangle = \sum_{k' \in \{k-\frac{d}{2}, \dots, k+\frac{d}{2}\}} A \exp\left(-\frac{\pi}{\sigma^2}(k'-k)^2\right) \underbrace{e^{i2\pi n(k'-k)/d}}_{|t_k\rangle}$$

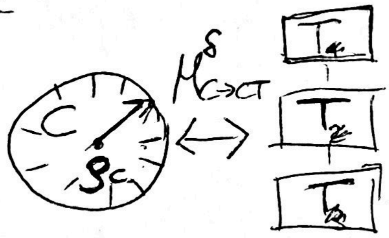
k : mean clock pointer position

σ : spread of clock pointer (for $\sigma \to 0 \leadsto |\psi(k)\rangle = |t_k\rangle$
for $\sigma = d \leadsto |\psi(k)\rangle = |E_n\rangle$

n : mean energy number ($E_n = \omega n$)

• / Bounds on the quasi-ideal clock's accuracy (Renner, Woods, Silva 2022)

Continuous/weak measurements to output information



} δ time step between two probes

← time registers: $|0\rangle$ no tick has occured

$|1\rangle$ a tick has occured

Recap on continous measurements / Quantum trajectories

$$\rho \Rightarrow \rho \otimes \underbrace{|\varphi\rangle\langle\varphi|}_{\text{Probe}} \longrightarrow \underset{\substack{\uparrow \\ \text{Interaction Syst-Probe}}}{U} \rho \otimes |\varphi\rangle\langle\varphi| U^{+} \xrightarrow[\substack{\text{Measure Probe} \\ \text{and find} \\ \text{state } |i\rangle}]{} \underbrace{\langle i|U|\varphi\rangle}_{F_i} \rho \underbrace{\langle\varphi|U^{+}|i\rangle}_{F_i^{+}} \otimes |i\rangle\langle i|$$

$$\rho \Rightarrow \frac{F_i \rho F_i^{+}}{\mathrm{Tr}(\text{°/.})}, \quad \sum_i F_i^{+} F_i = \mathbb{1} \Rightarrow \{F_i\} \text{ POVM}$$

~~If the outcome~~

$$\rho_C \longmapsto \mathcal{M}^{\delta}(\rho_C) = \sum_i M_i \rho_C M^{+} \otimes |0\rangle\langle 0| + \sum_i N_i \rho_C N_i^{+} \otimes |1\rangle\langle 1| \quad (\text{in general})$$

$$\left(= e^{-i\delta H_C} M_0^{\delta} \rho_C M_0^{\delta +} e^{i\delta H_C} \otimes |0\rangle\langle 0| + \sum_{j=0}^{d-1} e^{-i\delta H_C} M_{1,j}^{\delta} \rho_C M_{1,j}^{\delta} e^{i\delta H_C} \otimes |1\rangle\langle 1| \right)$$

Assume:
- reset clock

where $M_{1,j}^{\delta} = \sqrt{2\delta V_j}\, |\Psi_0\rangle\langle t_j|$ $\qquad |\Psi_0\rangle$: initial state of the clock

$M_0^{\delta} = \sqrt{1 - 2\delta V_C}$

$V_C = \sum_{i=0}^{d-1} V_i |t_i\rangle\langle t_i|$

Notation

$N \to d, \ |V_k\rangle \to |t_k\rangle$

$$= \rho_C \otimes |0\rangle\langle 0| - \delta\left(i[H_C, \rho_C] + \{V_C, \rho_C\}\right) \otimes |0\rangle\langle 0| + \delta \sum_i 2V_i |\Psi_0\rangle\langle t_i| \rho_C |t_i\rangle\langle\Psi_0| \otimes |1\rangle\langle 1|$$

$$P^{(1)}(t) = \lim_{\delta\to 0} \frac{tr_{CT}(\,|1\rangle\langle 1|\; M^{\delta}(\rho_c(t))\,)}{\delta} = 2\, tr_c(\,\underbrace{\Sigma V_i |t_i\rangle\langle t_i|}_{V_c}\,\rho_c(t)\,)$$

Clock evolution alone before the first tick

$|\psi_t\rangle = e^{-iH_c t - iV_c}|\psi_0\rangle$ i.e. information output via complex potential V_c

Results

the accuracy $R = \left(\frac{\tau}{\Delta\tau}\right)^2$ τ: mean time between ticks
$\Delta\tau$: ~~Variance~~ of time betw. ticks

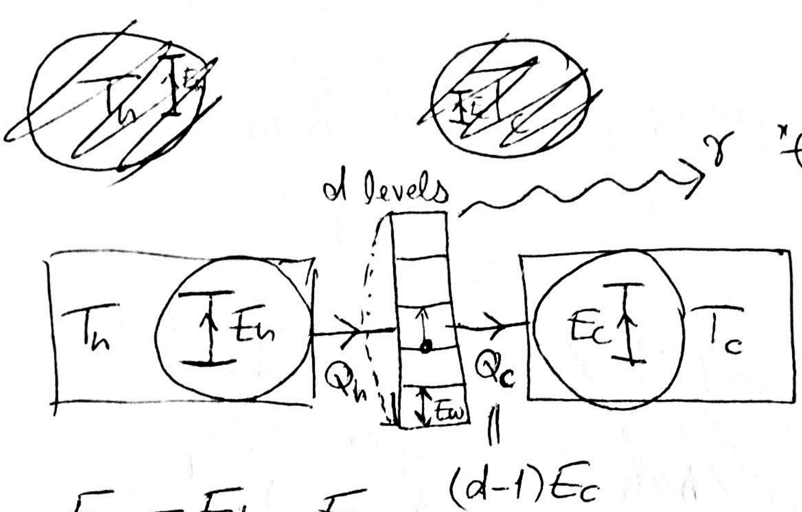
- of any classical clock is upperbounded by its dimension
  $R < d$
- of the precise (quantum) clock considered here is at least
  $R \geq d^{2-\varepsilon}$ for any $0 < \varepsilon \leq 1$

- Practical implications

Current ~~Ato~~ Optical atomic clocks achieve $R = O(10^{16})$ and if only measured over one hour $R \sim 10^{18}$

$\Rightarrow$ Dimension must be at least $\sim 10^{10} \sim 2^{33}$
i.e. we need 33 perfectly controlable qbits to attain this limit at current accuracy.

°/ Optional: A (classical) Thermodyn. clock (~~Erker~~ et al, 2017)



γ "tick" with $E_\gamma = (d-1)E_\omega$

$$Q_c = (d-1)E_c$$
$$Q_h = (d-1)E_h$$
$$(\beta_c-\beta_h)Q_c - \beta_h \underbrace{E_\gamma}_{Q_h-Q_c} = \beta_c Q_c - \beta_h Q_h$$

$$E_\omega = E_h - E_c$$

$$H_0 = \sum_{j=h,c} E_j |1\rangle\langle 1| + \sum_{k=0}^{d-1} k\, E_\omega |k\rangle\langle k|_\omega$$

$$H_{int} = \underset{\substack{\uparrow \\ \text{small}}}{g} \sum_{k=0}^{d-1} |1\rangle_c |0\rangle_h |k+1\rangle_\omega \langle 0|_c \langle 1|_h \langle k|_\omega + h.c.$$

bias the transition into this direction

by $\frac{E_h}{T_h} < \frac{E_c}{T_c}$

$$\text{Prob}(|0\rangle_c|1\rangle_h) \sim e^{-\beta_h E_h} \equiv p_1 > p_0 \equiv e^{-\beta_c E_c} \sim \text{Prob}(|1\rangle_c|0\rangle_h)$$

Small coupling $g \rightsquigarrow$ biased random walk with spead

$$p_1 - p_0 = v$$

$$\tau = \frac{d}{p_1 - p_0} \quad \text{(average time for a tick)}$$

$$\cancel{\Rightarrow} \quad \frac{d}{dt}\sigma^2(t) = p_1 + p_0 \rightsquigarrow \underbrace{\cancel{(\Delta\tau)}}^{\sigma^2(\tau)} = d\,\frac{p_1+p_0}{p_1-p_0}, \quad (\Delta\tau)^2 = \frac{\sigma^2(\tau)}{v}$$

heat dissipated

$$R = \left(\frac{\tau}{\Delta\tau}\right)^2 = d\left(\frac{p_0-p_1}{p_0+p_1}\right) = d\tanh\left(\frac{(\beta_c-\beta_h)Q_c - \beta_h E_\gamma}{2d}\right)$$

$$\xrightarrow{Q_c\to\infty} d \quad \text{(optimal classical clock)}$$

$$R \xrightarrow{d\to\infty} \frac{\beta_c Q_c - \beta_h Q_h}{2} = \frac{\Delta S}{2} \leftarrow \text{Entropy}$$

"to measure how time passes, one needs to increase the entropy"

$$\frac{\frac{1}{\sqrt{p_0p_1}}\,p_0 - p_1}{\frac{1}{\sqrt{p_0p_1}}\,p_0 + p_1} = \cancel{\frac{e^{-\beta_c E_c} - e^{-\beta_h E_h}}{e^{-\beta_c E_c} + e^{-\beta_h E_h}}} = \frac{\sqrt{\frac{p_0}{p_1}} - \sqrt{\frac{p_1}{p_0}}}{\sqrt{\frac{p_0}{p_1}} - \sqrt{\frac{p_1}{p_0}}}$$

$$\sqrt{\frac{p_0}{p_1}} = e^{(\beta_c E_c - \beta_h E_h)/2}$$